

# Poisson-Lie dynamical $r$ -matrices from Dirac reduction

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## Abstract

The Dirac reduction technique used previously to obtain solutions of the classical dynamical Yang-Baxter equation on the dual of a Lie algebra is extended to the Poisson-Lie case and is shown to yield naturally certain dynamical  $r$ -matrices on the duals of Poisson-Lie groups found by Etingof, Enriquez and Marshall in math.QA/0403283.

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# 1 Introduction

Let  $G$  be a connected Poisson-Lie (PL) group of the coboundary type, and denote by  $\mathcal{G} := \text{Lie}(G)$  its Lie algebra. The Poisson bracket (PB) on  $G$  can be encoded by the formula

$$\{g_1, g_2\}_G = [g_1 g_2, R], \quad g \in G, \quad (1.1)$$

where  $R \in \mathcal{G} \wedge \mathcal{G}$  solves the (modified) classical Yang-Baxter equation

$$[R_{12}, R_{13}] + [R_{12}, R_{23}] + [R_{13}, R_{23}] = \mathcal{I}_R, \quad (1.2)$$

with some  $G$ -invariant  $\mathcal{I}_R \in \mathcal{G} \wedge \mathcal{G} \wedge \mathcal{G}$ . Consider a PL subgroup  $K \subseteq G$  and the corresponding dual PL group  $K^*$ . Fix some open submanifold  $\check{K}^* \subset K^*$ . By definition, a *PL dynamical  $r$ -matrix with respect to the pair  $K \subseteq G$*  is an ‘admissible’ (smooth or meromorphic) mapping  $r : \check{K}^* \rightarrow \mathcal{G} \wedge \mathcal{G}$ , which is  $\mathcal{K}$ -equivariant in the natural sense and satisfies the equation

$$[R_{12} + r_{12}, R_{23} + r_{23}] + K_1^a \mathcal{L}_{K_a} r_{23} + \text{cycl.perm.} = \mathcal{I}_{R,r}, \quad (1.3)$$

where  $\{K^a\} \subset \mathcal{K} := \text{Lie}(K)$ ,  $\{K_a\} \subset \mathcal{K}^* = \text{Lie}(K^*)$  are bases in duality,  $\mathcal{I}_{R,r} \in \mathcal{G} \wedge \mathcal{G} \wedge \mathcal{G}$  is a  $G$ -invariant constant, and  $\mathcal{L}_{K_a}$  is the left-derivative associated with  $K_a \in \mathcal{K}^*$ . Equation (1.3) is called the PL-CDYBE for the pair  $K \subseteq G$ . (The shorthand CDYBE stands for ‘classical dynamical Yang-Baxter equation’.) Motivated by the study of the PL symmetries of the chiral Wess-Zumino-Novikov-Witten phase space [1], this equation was considered for  $K = G$  in [2, 3]. The general case of a proper PL subgroup  $K \subset G$  (also without restricting  $G$  to be of the coboundary type) was investigated in [4]. See also [5, 6] for an even more general notion of PL dynamical  $r$ -matrices. If  $R$  is set to zero, then any Lie subgroup  $K \subset G$  is a PL subgroup, and the dual  $K^*$  becomes  $\mathcal{K}^*$  with its linear Lie-Poisson structure. Thus for  $R = 0$  the PL-CDYBE reproduces the CDYBE for the pair  $\mathcal{K} \subseteq \mathcal{G}$  as defined in [7].

Etingof and Varchenko [7] introduced a useful technique of reduction of variables that connects, for example, solutions of the CDYBE on  $\mathcal{G}^*$  and on  $\mathcal{K}^*$ , where  $\mathcal{K}$  is a Levi subalgebra of a simple Lie algebra  $\mathcal{G}$ . In [8] the reduction technique of [7] was shown to be equivalent to the application of a suitable Dirac reduction to the PL groupoid that underlies the geometric interpretation of the CDYBE. The reduction technique of [7] (see also [9]) has been generalized in [4] to the PL case, leading to new PL dynamical  $r$ -matrices.

The purpose of the present note is to show that, as was anticipated in [2], the Dirac reduction method of [8] extends naturally to the PL case, too. This method permits us to obtain a better understanding of some constructions in [4], and it may prove useful in future investigations as well.

For simplicity, we shall focus on the *triangular* PL dynamical  $r$ -matrices, which satisfy the extra condition  $\mathcal{I}_{R,r} = \mathcal{I}_R$  by definition. In our Dirac reduction the starting phase space will be the manifold  $G \times \check{K}^*$  equipped with a PB encoding a triangular PL dynamical  $r$ -matrix  $r : \check{K}^* \rightarrow \mathcal{G} \wedge \mathcal{G}$ . If  $H \subset K$  is a PL subgroup and certain further conditions are satisfied, then Dirac reduction yields  $G \times \check{H}^*$  in such a way that the reduced PB (the ‘Dirac bracket’) encodes another triangular PL dynamical  $r$ -matrix  $r^* : \check{H}^* \rightarrow \mathcal{G} \wedge \mathcal{G}$ . In particular, if the starting  $r$ -matrix is zero, then we recover the  $\sigma_{\mathcal{H}}^{\mathcal{G}}$  family of  $r$ -matrices discovered in [4]. The

conditions guaranteeing our Dirac reduction to work are also used in the direct definition of  $\sigma_{\mathcal{H}}^{\mathcal{G}}$  in [4]. Here the conditions will be seen to emerge naturally from the construction.

The main part of the paper is Section 3, where we deal with the Dirac reduction of  $K^*$  to  $\check{H}^*$  and its application to the triangular solutions of (1.3). The general (non-triangular) case, together with examples and their possible applications, is briefly discussed in Section 4.

## 2 Geometric model for triangular dynamical $r$ -matrices

The PL-CDYBE is encoded by the Jacobi identities of the PB on certain Poisson manifolds. The following model, valid in the triangular case, can be found in [4].

Below, for any Lie group  $A$  the adjoint action of  $a \in A$  on  $X \in \mathcal{A} := \text{Lie}(A)$  is denoted simply by  $\text{Ad}_a(X) = aXa^{-1}$ . In the same spirit, regarding  $a \in A$  as a matrix, we may write the left and right derivatives as  $\mathcal{L}_X a = Xa$ ,  $\mathcal{R}_X a = aX$  and so on. For an arbitrary function  $f$  on  $A$ , we have  $(\mathcal{L}_X f)(a) := \left. \frac{d}{dt} f(e^{tX}a) \right|_{t=0}$  and  $(\mathcal{R}_X f)(a)$  is defined similarly. Correspondingly, the  $\mathcal{A}^*$ -valued left and right ‘gradients’ are defined by

$$\langle \nabla_a f, X \rangle = (\mathcal{L}_X f)(a), \quad \langle \nabla'_a f, X \rangle = (\mathcal{R}_X f)(a). \quad (2.1)$$

We need to recall (see, e.g., [10]) that the PB on the dual  $K^*$  of a PL group  $K$  can be written as

$$\{f_1, f_2\}_{K^*}(\kappa) = \langle \nabla_{\kappa} f_1, \kappa(\nabla'_{\kappa} f_2) \kappa^{-1} \rangle. \quad (2.2)$$

Here  $\nabla_{\kappa} f_i, \nabla'_{\kappa} f_i \in \mathcal{K} = (\mathcal{K}^*)^*$ ,  $\langle \langle \cdot, \cdot \rangle \rangle$  is the ‘scalar product’ on the Drinfeld double Lie algebra  $\mathcal{D}(\mathcal{K}, \mathcal{K}^*)$ , the adjoint action of  $\kappa \in K^*$  on  $X \in \mathcal{K}$  refers to the Drinfeld double Lie group  $D(K, K^*)$  that contains  $K$  and  $K^*$  as Lie subgroups. We need also the infinitesimal left dressing action of  $\mathcal{K}$  on  $K^*$ , which is defined by the formula

$$\text{dress}_X \kappa = \kappa(\kappa^{-1} X \kappa)_{\mathcal{K}^*}, \quad \forall X \in \mathcal{K}, \quad (2.3)$$

where we use the decomposition of  $\forall Y \in \mathcal{D}(\mathcal{K}, \mathcal{K}^*)$  into  $Y = Y_{\mathcal{K}} + Y_{\mathcal{K}^*}$  with  $Y_{\mathcal{K}} \in \mathcal{K}$ ,  $Y_{\mathcal{K}^*} \in \mathcal{K}^*$ .

Fixing  $\check{K}^*$  to be an open submanifold of  $K^*$ , consider the manifold

$$Q := G \times \check{K}^* = \{(g, \kappa)\}. \quad (2.4)$$

We write  $Q(\check{K}^*)$  if we want to emphasize the dependence on the choice of  $\check{K}^*$ . For functions  $\phi$  on  $G$  and  $f$  on  $\check{K}^*$ , let  $\phi'$  and  $\hat{f}$  be the functions on  $Q$  given by

$$\phi'(g, \kappa) = \phi(g), \quad \hat{f}(g, \kappa) = f(\kappa). \quad (2.5)$$

Take an admissible function

$$r : \check{K}^* \rightarrow \mathcal{G} \wedge \mathcal{G} \quad (2.6)$$

and try to define a PB on  $Q$  by means of the ansatz

$$\{\hat{f}_1, \hat{f}_2\}_Q(g, \kappa) = \{f_1, f_2\}_{\check{K}^*}(\kappa), \quad (2.7)$$

$$\{\phi', \hat{f}\}_Q(g, \kappa) = \langle \nabla'_g \phi, \nabla_{\kappa} f \rangle, \quad (2.8)$$

$$\{\phi'_1, \phi'_2\}_Q(g, \kappa) = \langle \nabla'_g \phi_1 \otimes \nabla'_g \phi_2, R + r(\kappa) \rangle - \langle \nabla_g \phi_1 \otimes \nabla_g \phi_2, R \rangle, \quad (2.9)$$

where  $R$  is the underlying constant solution of (1.2). One can verify

**Proposition 2.1.** *The bracket  $\{ , \}_Q$  satisfies the Jacobi identity if and only if the infinitesimal equivariance condition*

$$\text{dress}_X r = [X \otimes 1 + 1 \otimes X, r] \quad \forall X \in \mathcal{K}, \quad (2.10)$$

and the PL-CDYBE (1.3) with  $\mathcal{I}_{R,r} = \mathcal{I}_R$  hold for  $r$ .

### 3 Dirac reduction and dynamical $r$ -matrices

Let  $H \subset K \subseteq G$  be a chain of connected PL subgroups of  $G$ . Given a Poisson manifold  $(Q(\check{K}^*), \{ , \}_{Q(\check{K}^*)})$ , we wish to reduce it to a Poisson manifold of the same kind, but with respect to the subgroup  $H \subset K$ . We wish to achieve this by viewing  $Q(\check{H}^*)$  as a submanifold of  $Q(\check{K}^*)$  specified by *second class* constraints in Dirac's sense [11]. Crucially, the constraints must be such that the reduced PB (the 'Dirac bracket') resulting from  $\{ , \}_{Q(\check{K}^*)}$  should have the form of  $\{ , \}_{Q(\check{H}^*)}$ . If this happens, then the triangular  $r$ -matrix  $r : \check{K}^* \rightarrow \mathcal{G} \wedge \mathcal{G}$  contained in  $\{ , \}_{Q(\check{K}^*)}$  gives rise to a reduced triangular  $r$ -matrix  $r^* : \check{H}^* \rightarrow \mathcal{G} \wedge \mathcal{G}$  contained in  $\{ , \}_{Q(\check{H}^*)}$ .

#### 3.1 Dirac reduction of $K^*$ to $\check{H}^*$

The reduction of  $r$ -matrices sketched above can only work if an open submanifold of  $(H^*, \{ , \}_{H^*})$  can be obtained as the Dirac reduction of  $(K^*, \{ , \}_{K^*})$ . To investigate the condition for this, let  $\mathcal{D}(\mathcal{K}, \mathcal{K}^*)$  and  $\mathcal{D}(\mathcal{H}, \mathcal{H}^*)$  be the Drinfeld doubles of the Lie bialgebras corresponding to the PL groups  $K$  and  $H$ . As linear spaces,

$$\mathcal{D}(\mathcal{K}, \mathcal{K}^*) = \mathcal{K} + \mathcal{K}^*, \quad \mathcal{D}(\mathcal{H}, \mathcal{H}^*) = \mathcal{H} + \mathcal{H}^*, \quad (3.1)$$

where  $\mathcal{H} = \text{Lie}(H)$ ,  $\mathcal{H}^* = \text{Lie}(H^*)$  and similarly for  $K$ .

We have assumed that  $H \subset K$  is a connected PL subgroup, and this is known [12] to be equivalent to the condition that  $\mathcal{H}^\perp \subset \mathcal{K}^*$ ,

$$\mathcal{H}^\perp = \{ \alpha \in \mathcal{K}^* \mid \langle \alpha, X \rangle = 0 \quad \forall X \in \mathcal{H} \}, \quad (3.2)$$

is an ideal of the Lie subalgebra  $\mathcal{K}^* \subset \mathcal{D}(\mathcal{K}, \mathcal{K}^*)$ . Next,  $H^*$  must clearly be a Lie subgroup of  $K^*$  for our construction, and this requires that  $\mathcal{H}^* \subset \mathcal{K}^*$  must be a Lie subalgebra. We can encode these data in a vector space decomposition

$$\mathcal{K} = \mathcal{H} + \mathcal{M}, \quad (3.3)$$

which induces

$$\mathcal{K}^* = \mathcal{H}^* + \mathcal{M}^* \quad \text{with} \quad \mathcal{H}^* = \mathcal{M}^\perp, \quad \mathcal{M}^* = \mathcal{H}^\perp. \quad (3.4)$$

In addition to  $\mathcal{H}^*$  being a Lie subalgebra and  $\mathcal{M}^*$  being a Lie ideal, we shall need (see also Remark 3.5 below) the decomposition (3.3) to be reductive

$$[\mathcal{H}, \mathcal{M}] \subset \mathcal{M}, \quad (3.5)$$

and of course the constraints specifying  $\check{H}^*$  inside  $K^*$  must be second class. Let  $\{M^i\} \subset \mathcal{M}$  be a basis. The second class nature of the constraints turns out equivalent to the non-degeneracy of the matrix

$$C^{ij}(\lambda) = \langle\langle (\lambda M^i \lambda^{-1})_{\mathcal{M}}, \lambda M^j \lambda^{-1} \rangle\rangle \quad \text{for } \lambda \in \check{H}^*, \quad (3.6)$$

defined using the Drinfeld double  $D(K, K^*)$ .

We next show that (3.6) together with the foregoing other assumptions guarantees the desired reduction of  $K^*$  to  $\check{H}^*$ . We begin by proving some auxiliary statements.

**Lemma 3.1.** *With the above notations, suppose that  $\mathcal{H} \subset \mathcal{K}$  and  $\mathcal{H}^* \subset \mathcal{K}^*$  are Lie subalgebras,  $\mathcal{H}^\perp \subset \mathcal{K}^*$  is a Lie ideal and  $[\mathcal{H}, \mathcal{M}] \subset \mathcal{M}$ . Then  $\mathcal{H} + \mathcal{H}^*$  is a Lie subalgebra of the double  $\mathcal{D}(\mathcal{K}, \mathcal{K}^*)$ . This subalgebra of  $\mathcal{D}(\mathcal{K}, \mathcal{K}^*)$  can be identified with the double  $\mathcal{D}(\mathcal{H}, \mathcal{H}^*)$ .*

*Proof.* We need to show that  $[\mathcal{H}, \mathcal{H}^*] \subset \mathcal{H} + \mathcal{H}^*$  inside  $\mathcal{D}(\mathcal{K}, \mathcal{K}^*)$ . With the Lie bracket  $[\ , \ ]$  and invariant ‘scalar product’  $\langle\langle \ , \ \rangle\rangle$  of  $\mathcal{D}(\mathcal{K}, \mathcal{K}^*)$ , we have

$$\langle\langle [\mathcal{H}^*, \mathcal{H}], \mathcal{M} \rangle\rangle = \langle\langle \mathcal{H}^*, [\mathcal{H}, \mathcal{M}] \rangle\rangle \subset \langle\langle \mathcal{H}^*, \mathcal{M} \rangle\rangle = \{0\},$$

since  $[\mathcal{H}, \mathcal{M}] \subset \mathcal{M}$ , and

$$\langle\langle [\mathcal{H}, \mathcal{H}^*], \mathcal{M}^* \rangle\rangle = \langle\langle \mathcal{H}, [\mathcal{H}^*, \mathcal{M}^*] \rangle\rangle \subset \langle\langle \mathcal{H}, \mathcal{M}^* \rangle\rangle = \{0\},$$

since  $[\mathcal{H}^*, \mathcal{M}^*] \subset \mathcal{M}^*$  as  $\mathcal{M}^* = \mathcal{H}^\perp$ . *Q.E.D.*

**Lemma 3.2.** *Under the assumptions of Lemma 3.1, consider the connected Lie subgroup  $H^* \subset K^*$  corresponding to  $\mathcal{H}^* \subset \mathcal{K}^*$ . Parametrize the elements in some neighbourhood of  $H^*$  in  $K^*$  as*

$$\kappa = \lambda e^\mu \quad \lambda \in H^*, \quad \mu \in \check{\mathcal{M}}^*, \quad (3.7)$$

where  $\check{\mathcal{M}}^*$  is some neighbourhood of zero in  $\mathcal{M}^*$ . Take a function  $F \in \mathcal{F}(H^*)$  and extend it (locally) to  $f \in \mathcal{F}(K^*)$  by

$$f(\lambda e^\mu) = F(\lambda). \quad (3.8)$$

Then

$$\nabla_\lambda f = \nabla_\lambda F \quad \text{and} \quad \nabla'_\lambda f = \nabla'_\lambda F. \quad (3.9)$$

*Proof.* In principle,  $\nabla_\lambda f \in \mathcal{K}$  and  $\nabla_\lambda F \in \mathcal{H}$ . For  $X \in \mathcal{H}^*$  we have  $f(e^{tX}\lambda) = F(e^{tX}\lambda)$ , and for  $Y \in \mathcal{M}^*$  we have  $f(e^{tY}\lambda) = f(\lambda \lambda^{-1} e^{tY} \lambda) = f(\lambda e^{t\lambda^{-1}Y\lambda}) = F(\lambda)$  since  $\lambda^{-1}Y\lambda \in \mathcal{M}^*$  by  $[\mathcal{H}^*, \mathcal{M}^*] \subset \mathcal{M}^*$ . This implies the first equality in (3.9). The second equality follows similarly, and actually it is also a consequence of the first one. Indeed,  $\nabla'_\lambda F = (\lambda^{-1} \nabla_\lambda F \lambda)_\mathcal{H}$  in the double of  $\mathcal{H}$ , and  $\nabla'_\lambda f = (\lambda^{-1} \nabla_\lambda f \lambda)_\mathcal{K}$  in the double of  $\mathcal{K}$  on general grounds, which implies the second equality by Lemma 3.1. *Q.E.D.*

**Lemma 3.3.** *Keeping the preceding assumptions, for a constant  $M \in \mathcal{M}$  define the function  $\xi_M$  on a neighbourhood of  $H^*$  in  $K^*$  by*

$$\xi_M(\lambda e^\mu) := \langle \mu, M \rangle \quad (\lambda \in H^*, \mu \in \check{\mathcal{M}}^*). \quad (3.10)$$

*For this function, we have*

$$\nabla'_\lambda \xi_M = M, \quad \nabla_\lambda \xi_M = (\lambda M \lambda^{-1})_{\mathcal{K}} = (\lambda M \lambda^{-1})_{\mathcal{M}}. \quad (3.11)$$

*As a consequence,*

$$\{f, \xi_M\}_{K^*}(\lambda) = 0 \quad (\forall \lambda \in H^*) \quad (3.12)$$

*for any functions  $f$  and  $\xi_M$  defined in (3.8), (3.10).*

*Proof.* It is simple to confirm  $\nabla'_\lambda \xi_M = M$  directly from the definition, and this implies  $\nabla_\lambda \xi_M = (\lambda M \lambda^{-1})_{\mathcal{K}}$  by the universal connection between left and right derivatives. The last equality in (3.11) follows since  $\langle \langle \lambda \mathcal{M} \lambda^{-1}, \mathcal{H}^* \rangle \rangle = \langle \langle \mathcal{M}, \lambda^{-1} \mathcal{H}^* \lambda \rangle \rangle \subset \langle \langle \mathcal{M}, \mathcal{H}^* \rangle \rangle = \{0\}$ . By using (2.2), the statement of (3.12) is a consequence of the fact that  $\nabla_\lambda f \in \mathcal{H}$  and  $\nabla'_\lambda \xi_M \in \mathcal{M}$ . Indeed,  $\langle \langle \mathcal{H}, \lambda \mathcal{M} \lambda^{-1} \rangle \rangle = \langle \langle \lambda^{-1} \mathcal{H} \lambda, \mathcal{M} \rangle \rangle \subset \langle \langle \mathcal{H} + \mathcal{H}^*, \mathcal{M} \rangle \rangle = \{0\}$  by Lemma 3.1. *Q.E.D.*

We are now ready to prove our main auxiliary statement.

**Theorem 3.4.** *Let us adopt the assumptions of Lemma 3.1, and consider a submanifold  $\check{H}^* \subset K^*$  defined locally by the constraints  $\xi_{M^i} = 0$ , where the functions  $\xi_{M^i}$  are associated by (3.10) with a basis  $\{M^i\}$  of  $\mathcal{M}$ . Then the PBs of the constraints are given by*

$$C^{ij}(\lambda) := \{\xi_{M^i}, \xi_{M^j}\}_{K^*}(\lambda) = \langle \langle (\lambda M^j \lambda^{-1})_{\mathcal{M}^*}, (\lambda M^i \lambda^{-1})_{\mathcal{M}} \rangle \rangle \quad (\lambda \in \check{H}^*). \quad (3.13)$$

*If the matrix  $C^{ij}(\lambda)$  is non-degenerate for  $\lambda \in \check{H}^*$ , then the Dirac reduction of  $(K^*, \{ , \}_{K^*})$  yields  $(\check{H}^*, \{ , \}_{\check{H}^*})$ .*

*Proof.* Let the functions  $F_n$  and  $f_n$  be related by (3.8) for  $n = 1, 2$ . The statement of the theorem follows by combining the preceding lemmas with the standard formula [11] of the Dirac bracket,  $\{ , \}^*$ :

$$\{F_1, F_2\}^*(\lambda) = \{f_1, f_2\}_{K^*}(\lambda) - \sum_{i,j} \{f_1, \xi_{M^i}\}_{K^*}(\lambda) (C^{-1}(\lambda))_{ij} \{\xi_{M^j}, f_2\}_{K^*}(\lambda).$$

The second term vanishes by (3.12), and the first term yields  $\{F_1, F_2\}_{H^*}$  on account of Lemmas 3.1 and 3.2. *Q.E.D.*

*Remark 3.5.* It is clear from the proof of Theorem 3.4 that the above used assumptions are not only sufficient, but also necessary for the desired Dirac reduction to work. For example, the assumption (3.5) is crucial in the proof of Lemma 3.1 on which Theorem 3.4 relies; we were led to this assumption in the  $R = 0$  case studied in [8], too. The same assumptions appear in the construction of PL dynamical  $r$ -matrices given in [4]. In a sense, Dirac reductions provides (for us) an explanation of these assumptions.

### 3.2 PL dynamical $r$ -matrices from Dirac reduction

By using the framework developed so far, the following result is essentially obvious.

**Theorem 3.6.** *Consider  $(Q(\check{K}^*), \{ , \}_{Q(\check{K}^*)})$  with the PB defined by a (possibly zero) triangular PL dynamical  $r$ -matrix  $r : \check{K}^* \rightarrow \mathcal{G} \wedge \mathcal{G}$ . Adopt the assumptions of Lemma 3.1 and suppose that  $C^{ij}(\lambda)$  (3.13) gives a non-degenerate matrix function on a non-empty submanifold  $\check{H}^* \subset H^*$ , which is contained in  $\check{K}^*$ . Then the submanifold  $Q(\check{H}^*) \subset Q(\check{K}^*)$  is defined by second class constraints, and the resulting Dirac bracket is of the type  $\{ , \}_{Q(\check{H}^*)}$  with the reduced  $r$ -matrix*

$$r^*(\lambda) = r(\lambda) + \rho(\lambda) \quad (\lambda \in \check{H}^*), \quad (3.14)$$

where  $\rho : \check{H}^* \rightarrow \mathcal{M} \wedge \mathcal{M} \subset \mathcal{K} \wedge \mathcal{K} \subset \mathcal{G} \wedge \mathcal{G}$  is given by

$$\rho(\lambda) = \sum_{i,j} (C^{-1}(\lambda))_{ij} (\lambda M^i \lambda^{-1})_{\mathcal{M}} \otimes (\lambda M^j \lambda^{-1})_{\mathcal{M}}, \quad \forall \lambda \in \check{H}^*. \quad (3.15)$$

Here,  $\{M^i\}$  is a basis of  $\mathcal{M}$  (3.3) and  $\text{Ad}_\lambda M^i = \lambda M^i \lambda^{-1}$  is defined using the double  $D(K, K^*)$ .

*Proof.* One can easily calculate the Dirac bracket similarly to the proof of Theorem 3.4. *Q.E.D.*

**Corollary 3.7.** (by Proposition 2.1). *Let  $r : \check{K}^* \rightarrow \mathcal{G} \wedge \mathcal{G}$  be a (possibly zero) triangular PL dynamical  $r$ -matrix for  $K \subseteq G$ . Suppose that the assumptions of Lemma 3.1 hold and  $C^{ij}$  (3.13) defines a non-degenerate matrix function on a non-empty submanifold,  $\check{H}^*$ , of  $H^* \cap \check{K}^*$ . Then  $r^* : \check{H}^* \rightarrow \mathcal{G} \wedge \mathcal{G}$  (3.14) gives a triangular PL dynamical  $r$ -matrix for  $H \subset G$ .*

If the bases  $\{M_i\} \subset \mathcal{M}^*$  and  $\{M^i\} \subset \mathcal{M}$  are in duality, then so are the bases  $\{\lambda M_i \lambda^{-1}\} \subset \mathcal{M}^*$  and  $\{(\lambda M^i \lambda^{-1})_{\mathcal{M}}\} \subset \mathcal{M}$  for any  $\lambda \in \check{H}^*$ . By the invertibility of  $C^{ij}(\lambda)$ ,  $\{(\lambda M^i \lambda^{-1})_{\mathcal{M}^*}\} \subset \mathcal{M}^*$  forms a basis, too. It follows that for any base element  $M_i \in \mathcal{M}^*$  and  $\lambda \in \check{H}^*$  there exists a unique element  $N_i(\lambda)$  that satisfies

$$\lambda^{-1} M_i \lambda = (\lambda^{-1} N_i(\lambda) \lambda)_{\mathcal{M}^*}, \quad N_i(\lambda) \in \mathcal{M}. \quad (3.16)$$

**Lemma 3.8.** *By using  $N_i(\lambda)$  (3.16), the triangular PL  $r$ -matrix in (3.15) can be written as*

$$\rho(\lambda) = - \sum_i N_i(\lambda) \otimes M^i = \sum_i M^i \otimes N_i(\lambda) \quad \forall \lambda \in \check{H}^*. \quad (3.17)$$

*Proof.* We have to show that the operator  $\hat{\rho}(\lambda) \in \text{End}(\mathcal{M}^*, \mathcal{M})$ , defined by

$$\hat{\rho}(\lambda)(M_k) = \sum_{i,j} (C^{-1}(\lambda))_{ij} (\lambda M^i \lambda^{-1})_{\mathcal{M}} \langle M_k, (\lambda M^j \lambda^{-1})_{\mathcal{M}} \rangle,$$

satisfies  $\hat{\rho}(\lambda)(M_k) = -N_k(\lambda)$ . By the definition of  $N_k(\lambda)$  and the invariance of the scalar product of  $\mathcal{D}(\mathcal{K}, \mathcal{K}^*)$ , we have

$$\hat{\rho}(\lambda)(M_k) = \sum_{i,j} (C^{-1}(\lambda))_{ij} (\lambda M^i \lambda^{-1})_{\mathcal{M}} \langle (\lambda^{-1} N_k(\lambda) \lambda)_{\mathcal{M}^*}, M^j \rangle$$

$$\begin{aligned}
&= \sum_{i,j} (C^{-1}(\lambda))_{ij} (\lambda M^i \lambda^{-1})_{\mathcal{M}} \langle \langle N_k(\lambda), \lambda M^j \lambda^{-1} \rangle_{\mathcal{M}^*} \rangle \\
&= \sum_{i,j,l} (C^{-1}(\lambda))_{ij} (\lambda M^i \lambda^{-1})_{\mathcal{M}} \langle \langle N_k(\lambda), \lambda M_l \lambda^{-1} \rangle \rangle C^{lj}(\lambda) = -N_k(\lambda),
\end{aligned}$$

as required. *Q.E.D.*

*Remark 3.9.* The dynamical  $r$ -matrix  $\rho$  in (3.15) is the same as  $\sigma_{\mathcal{H}}^{\mathcal{G}}$  found in [4]. In order to verify this, note that formula (3.17) implies the identity

$$\langle \langle (\lambda^{-1} u \lambda)_{\mathcal{M}}, \lambda^{-1} v \lambda \rangle \rangle = \sum_i \langle \langle (\lambda^{-1} u \lambda)_{\mathcal{M}}, \lambda^{-1} M^i \lambda \rangle \rangle \langle \langle (\lambda^{-1} v \lambda)_{\mathcal{M}}, \lambda^{-1} N_i(\lambda) \lambda \rangle \rangle$$

for all  $u, v \in \mathcal{M}$ ,  $\lambda \in \check{H}^*$ . According to [4] (property 1 above Theorem 2.2) this identity characterizes  $\sigma_{\mathcal{H}}^{\mathcal{G}}$  uniquely, if  $\sigma_H^{\mathcal{G}}$  is written in the form (3.17) *with some*  $N_i(\lambda)$ . The arguments that led to our Corollary 3.7 appear (for us) more enlightening than the direct proof of Theorem 2.2 in [4], which states that  $\sigma_{\mathcal{H}}^{\mathcal{G}}$  is a triangular PL dynamical  $r$ -matrix.

## 4 Discussion

It is important to note that the applicability of the Dirac reduction method is not restricted to the triangular case. In fact [2, 4], an arbitrary PL dynamical  $r$ -matrix  $r : \check{K}^* \rightarrow \mathcal{G} \wedge \mathcal{G}$  encodes a PB on the manifold

$$P = P(\check{K}^*) := \check{K}^* \times G \times \check{K}^* = \{(\tilde{\kappa}, g, \hat{\kappa})\}. \quad (4.1)$$

For admissible functions  $f \in \mathcal{F}(\check{K}^*)$  and  $\phi \in \mathcal{F}(G)$  one introduces  $\hat{f}, \tilde{f} \in \mathcal{F}(P)$  and  $\phi' \in \mathcal{F}(P)$  by  $\hat{f}(\tilde{\kappa}, g, \hat{\kappa}) = f(\hat{\kappa})$ ,  $\tilde{f}(\tilde{\kappa}, g, \hat{\kappa}) = f(\tilde{\kappa})$ ,  $\phi'(\tilde{\kappa}, g, \hat{\kappa}) = \phi(g)$ . One then postulates a bracket on the functions on  $P$  by the ansatz

$$\begin{aligned}
\{\hat{f}_1, \hat{f}_2\}_P(\tilde{\kappa}, g, \hat{\kappa}) &= \{f_1, f_2\}_{\check{K}^*}(\hat{\kappa}), & \{\tilde{f}_1, \tilde{f}_2\}_P(\tilde{\kappa}, g, \hat{\kappa}) &= -\{f_1, f_2\}_{\check{K}^*}(\tilde{\kappa}), \\
\{\phi', \hat{f}\}_P(\tilde{\kappa}, g, \hat{\kappa}) &= \langle \nabla'_g \phi, \nabla_{\hat{\kappa}} f \rangle, & \{\phi', \tilde{f}\}_P(\tilde{\kappa}, g, \hat{\kappa}) &= \langle \nabla_g \phi, \nabla_{\tilde{\kappa}} f \rangle, \\
\{\phi'_1, \phi'_2\}_P(\tilde{\kappa}, g, \hat{\kappa}) &= \langle \nabla'_g \phi_1 \otimes \nabla'_g \phi_2, R + r(\hat{\kappa}) \rangle - \langle \nabla_g \phi_1 \otimes \nabla_g \phi_2, R + r(\tilde{\kappa}) \rangle,
\end{aligned} \quad (4.2)$$

together with  $\{\hat{f}_1, \tilde{f}_2\}_P = 0$ , where  $f, f_i \in \mathcal{F}(\check{K}^*)$ ,  $\phi, \phi_i \in \mathcal{F}(G)$ ,  $\langle \cdot, \cdot \rangle$  denotes the canonical pairing between elements of  $\mathcal{G}^*$  and  $\mathcal{G}$ , and  $R$  is the chosen constant  $r$ -matrix (1.2). The ansatz (4.2) defines a PB if and only if the PL-CDYBE (1.3) and the equivariance condition (2.10) are valid for  $r$ . It is clear that under the assumptions of Theorem 3.4 the Dirac reduction of  $(P(\check{K}^*), \{ \cdot, \cdot \}_{P(\check{K}^*)})$  yields  $(P(\check{H}^*), \{ \cdot, \cdot \}_{P(\check{H}^*)})$ , and the accompanying reduction of the PL dynamical  $r$ -matrix is given by the same formula (3.14), (3.15) as in the triangular case. The content of this statement is precisely the ‘composition theorem’ (Theorem 2.7) of [4]. (Note also that  $\mathcal{I}_{R,r} = \mathcal{I}_{R,r^*}$  is easily checked by using (3.17).)

If  $R = 0$ , then the construction of dynamical  $r$ -matrices by Dirac reduction described above specializes to the construction given in [8]. This provides us with examples in the case



of an Abelian  $G^*$ . For non-Abelian  $G^*$  we do not know examples that are essentially different from those mentioned in [4]. If  $R$  is the standard (Drinfeld-Jimbo) *factorisable*  $r$ -matrix on a simple Lie algebra, then one can apply the reduction by taking  $\mathcal{K} = \mathcal{G}$  and taking  $\mathcal{H}$  to be a Levi (regular reductive) subalgebra of  $\mathcal{G}$ . Thus Corollary 3.7 yields triangular PL dynamical  $r$ -matrices for the Levi subgroups of  $G$ . The composition theorem can also be applied by taking the  $r_{\text{BFP}}$  solution [1] of the PL-CDYBE for  $K = G$  as the starting point [4]. Although not mentioned in [4], the same family of examples is available in the compact case as well, where a simple compact Lie group  $G$  is equipped with its standard PL structure and  $H \subset K = G$  is a regular reductive subgroup. (See also [3] for a description of  $r_{\text{BFP}}$  in PL terms.)

Incidentally, the Dirac reductions of  $r_{\text{BFP}}$  just alluded to can be seen as exchange  $r$ -matrices in the Wess-Zumino-Novikov-Witten model, obtained there by restricting the monodromy matrix to a regular reductive subgroup of  $G$ , i.e., by performing the corresponding Dirac reduction of the chiral WZNW PB defined by  $r_{\text{BFP}}$  [1]. The closely related trigonometric PL  $r$ -matrices of [6] can also be associated with suitable PL symmetries on the chiral WZNW phase space with restricted monodromy.

It appears an interesting open question whether one can relate the PL dynamical  $r$ -matrices to finite dimensional integrable systems by suitable extension of the constructions in [13, 14] to the PL case. Our basic idea for such generalization is to apply Hamiltonian reduction to  $(P(\tilde{K}^*), \{ , \}_{P(\tilde{K}^*)})$  by using the PL action of  $K$  generated by the PL momentum map  $\Lambda : P \rightarrow K^*$  given by

$$\Lambda : (\tilde{\kappa}, g, \hat{\kappa}) \mapsto \tilde{\kappa} \hat{\kappa}^{-1}. \quad (4.3)$$

In analogy with the constructions in [13, 14], a relevant reduction of  $P$  should be defined by setting the momentum map  $\Lambda$  to unity; in other words by imposing the first class constraints  $\hat{\kappa} = \tilde{\kappa}$ . However, we have not yet investigated how to obtain commuting Hamiltonians on the reduced phase space in this context. Naively, one expects to obtain such Hamiltonians from the functions of the form  $\phi'$  where  $\phi$  is a central function on  $G$ , but further work is required to see if this idea can really work or not.

It could be also interesting to develop the quantum version of our Dirac reduction algorithm. This may simplify the quantization of various dynamical  $r$ -matrices [9, 4].

**Acknowledgements.** This work was supported in part by the Hungarian Scientific Research Fund (OTKA) under grants T034170, T043159 and M036803. Thanks are due to J. Balog and I. Marshall for useful comments on the remark communicated here.

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